

# The gauge group in the real triad formulation of general relativity

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We construct explicitly generators of projectable four-dimensional diffeomorphisms and triad rotation gauge symmetries in a model of vacuum gravity where the fundamental dynamical variables in a Palatini formulation are taken to be a lapse, shift, densitized triad, extrinsic curvature, and the time-like components of the Ricci rotation coefficient. Time-foliation-altering diffeomorphisms are not by themselves projectable under the Legendre transformations. They must be accompanied by a metric- and triad-dependent triad rotation. The phase space on which these generators act includes all of the gauge variables of the model.

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## I. INTRODUCTION

General covariance is the fundamental symmetry of the classical Lagrangian formulation of general relativity. One might be tempted to think that it is destroyed in the Hamiltonian version of the theory in which, ostensibly, time is distinguished from space. In recent works we have established a general framework for analyzing and describing the preservation of local (gauge) symmetries under the Legendre map from configuration-velocity space (the tangent bundle) to phase space (the cotangent bundle) [1]. The program was applied to general relativity using conventional metric variables and to Einstein-Yang-Mills theory [2]. In the former case it was shown that the four-dimensional diffeomorphism group is not preserved under this map. Rather, infinitesimal elements of the gauge group contain a compulsory dependence on the lapse and shift functions. In the Einstein-Yang-Mills case gauge transformations also involve internal gauge transformations. Nevertheless there is a sense in which all diffeomorphisms act as transformations on the full phase space of the theory, since any such transformation may be realized with some metric plus Yang-Mills field.

In this paper we apply the program to a tetrad version of Einstein's general theory of relativity. Spinorial and tetrad formulations of general relativity were introduced

initially with the intention of coupling fermionic matter fields to gravity in an eventual quantum theory of gravity. The emerging Dirac-Bergmann constraint algorithm was applied to Lagrangian or Hamiltonian models by Heller and Bergmann [3], DeWitt and DeWitt [4], Dirac [5], and Schwinger [6]. In these early investigations an effort was not made to retain the local Lorentz freedom to rotate and boost the tetrad axes. Schwinger chose one of the vectors of the tetrad to point perpendicular to the equal-time hypersurfaces. The coordinate time was also taken over in the Hamiltonian model as the evolutionary time. Consequently, all that remained were local rotations tangential to the equal-time hypersurfaces. This will actually be our point of departure below. Later, several authors have been concerned with retaining the full local Lorentz group [7–13]. Generally, retention of the local Lorentz group is achieved by adding auxiliary pure gauge boost variables. We will not pursue this direction here, although our analysis can easily be generalized to include the full local Lorentz symmetry. Clayton analyzed the symmetry of triad models only under spatial diffeomorphisms [14].

Interest in time-foliation-conforming triads has surged recently. This interest stems from Ashtekar's approach to general relativity [15–17]. The relation of Ashtekar variables to triads was elucidated by Goldberg [18], Friedman

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and Jack [19], and Henneaux, Nelson, and Schomblond [11].

Our novel contribution in this paper to this extensive literature has to do with the gauge group stemming from the full four-dimensional diffeomorphism group with local triad rotations appended where necessary. We show that this group is retained under the Legendre map to phase space. This group acts as a transformation group on all of the dynamical variables of the theory (including pure gauge variables). In particular, time evolution is recognizable as a symmetry transformation on members of equivalence classes (under the full four-dimensional diffeomorphism group) of solutions of the dynamical equations. The implications for an eventual quantum theory of gravity and its associated problem of time, we feel, are profound: Any reasonable quantization procedure, whether it take as fundamental gravitational variables the metric, tetrad, or triad fields, must respect and presumably exploit this symmetry.

Our plan is as follows. Our focus is local; the analysis is strictly applicable to spatially compact spacetimes. We work within a 3+1 spacetime foliation; the fundamental fields in this model include triads, lapse functions, and shift vectors. Our dynamical treatment also includes some of the time-like terms of the Ricci rotation coefficients as independent fields in a Palatini-like variational approach. We show that the full diffeomorphism-induced gauge group is present, and we display explicitly the corresponding generators on the full phase space (which includes gauge variables).

After completing the Lagrangian and Hamiltonian descriptions in Section II, we review the general projectability requirements in Section III, deducing that coordinate transformations for which  $\delta x^0 \neq 0$  are not by themselves projectable. A gauge rotation constructed with the aid of the four-dimensional Ricci rotation coefficients must be appended to them. Next, in Section IV we determine the variations engendered by the secondary constraints. We demonstrate that it is possible to determine the structure coefficients of the gauge group algebra from knowledge of the action of the gauge group on configuration-velocity variables alone, without reference to Poisson brackets. Finally in Section V we display the full set of gauge-symmetry generators. In Section VI we summarize and discuss implications of the work and future extensions, such as to the Ashtekar formulation itself [20].

## II. THE TETRAD ACTION

We take as our fundamental dynamical variables the tetrad one form fields  $e_\mu^I$ , with inverse  $E_I^\mu$ , yielding the metric  $g_{\mu\nu} = e_\mu^I e_{I\nu}$ . The greek coordinate indices range from 0 to 3, as do the upper case latin orthonormal basis labels. The Minkowski index labels are raised and lowered with the Minkowski metric  $\eta^{IJ} = \eta_{IJ}$ , which we take to have signature  $-+++$ . Lower case latin indices

from the beginning of the alphabet will represent spatial coordinates, while lower case latin indices from the middle of the alphabet will denote orthonormal triad labels (and therefore may be raised or lowered using  $\delta_{ij}$ ).

The Ricci rotation coefficients  $\Omega_\mu^{IJ}$  provide a Lorentz-group connection which leaves the tetrad covariantly constant and are given by

$$\Omega_\mu^{IJ} = E^{\beta I} e_{\beta,\mu}^J - \Gamma_{\gamma\mu}^\beta E^{\gamma I} e_\beta^J, \quad (2.1)$$

where the  $\Gamma_{\gamma\mu}^\beta$  are the Christoffel symbols. The associated curvature is

$${}^4R_{\mu\nu}^{IJ} = 2\partial_{[\mu}\Omega_{\nu]}^{IJ} + 2\Omega_{[\mu}^{IM}\Omega_{\nu]M}^J.$$

A convenient action expressed in terms of these variables is one half the Hilbert action,

$$\frac{1}{2} \int d^4x \sqrt{|{}^4g|} {}^4R_{\mu\nu}^{IJ} E_I^\mu E_J^\nu. \quad (2.2)$$

We make a 3 + 1 decomposition of the tetrad in terms of triad  $T_i^a$ , lapse  $N$ , and shift  $N^a$  as follows:

$$E^\mu_I = \begin{pmatrix} N^{-1} & 0 \\ -N^{-1}N^a & T_i^a \end{pmatrix},$$

with inverse

$$e^I_\mu = \begin{pmatrix} N & 0 \\ t^i_a N^a & t^i_a \end{pmatrix},$$

where  $t^i_a T_i^b = \delta_b^a$ ,  $t^i_a T_j^a = \delta_j^i$ .

In a succeeding paper, we will be applying our technique to the Ashtekar formalism in full [20], and so at this point we adopt notation with that end in mind. We introduce as some of our fundamental dynamical configuration variables the components of a contravariant triad field with density one under spatial diffeomorphisms:

$$\tilde{T}_i^a := t T_i^a,$$

where  $t$  is the determinant of  $t^i_a$ . We also will represent densities of arbitrary positive or negative weight with appropriate numbers of over- or under-tildes, respectively. We also note that

$$\Omega_a^{0i} = T^{bi} K_{ba} =: K_a^i,$$

where  $K_{ab}$  is the extrinsic curvature.

The Lagrangian density expressed in terms of these new variables, after subtracting a total spatial divergence, is (where  $\cdot$  is partial with respect to time)

$$\begin{aligned} \mathcal{L} = & -K_a^i \tilde{T}_i^a + \epsilon^{ijk} \tilde{T}_i^a K_a^j \Omega_0^k - 2N^a \tilde{T}_i^b D_{[a} K_{b]}^i \\ & + \frac{1}{2} N \tilde{T}_i^a \tilde{T}_j^b ({}^3R_{ab}^{ij} + K_a^i K_b^j - K_b^i K_a^j). \end{aligned} \quad (2.3)$$

We translate from one triad label index to two index antisymmetric objects by taking the dual using the three-dimensional Levi-Civita symbol  $\epsilon^{ijk}$ ; for example,

$$\Omega_0^i = \frac{1}{2} \epsilon^{ijk} \Omega_0^{jk} .$$

In the relation above  $D_a$  denotes the covariant derivative formed with the three-dimensional rotation coefficients and Christoffel symbols, so we have, for example,

$$D_a T_i^b = \partial_a T_i^b + {}^3\Gamma_{ca}^b T_i^c + \omega_a^{ij} T_j^b = 0 ,$$

where the 3-dimensional rotation coefficients

$$\begin{aligned} \Omega_a^{ij} &= g_{ac} \tilde{T}^{b[i} \tilde{T}^{j]c}{}_{,b} + \tilde{T}^{b[i} \tilde{T}^{j]}{}_{,a} \tilde{T}_a^c{}_{,b} \\ &\quad + \tilde{T}_b^{[i} \tilde{T}^{j]b}{}_{,a} + \tilde{T}_c^c \tilde{T}_{k,b} \tilde{T}_a^{[i} \tilde{T}^{j]b} \\ &= : \omega_a^{ij} \end{aligned}$$

are constructed from the triad fields.

We will undertake a Palatini-type variation in which we vary  $\tilde{T}_i^a$ ,  $K_a^i$ ,  $N^a$ ,  $\Omega_0^i$ , and  $\tilde{N}$  independently. This choice of variables is justified because the variables  $K_a^i$  and  $\Omega_0^i$  can be considered as independent auxiliary variables for the Lagrangian (2.3): That is, their equations of motion allow their determination in terms of the other variables.

It may seem strange initially that we have identified  $\Omega_0^{ij}$  as an independent variable. Examination of its triad dependence in (2.1) reveals the rationale:

$$\begin{aligned} \Omega_0^{ij} &= \epsilon^{ijk} \Omega_0^k = T^{a[i} t_a^{j]} - g_{a\mu} \Gamma_{b0}^\mu T^{b[i} T^{j]a} \\ &= \tilde{T}^{a[i} \tilde{t}_a^{j]} - g_{a\mu} \Gamma_{b0}^\mu T^{b[i} T^{j]a} \\ &= \tilde{T}^{a[i} \tilde{t}_a^{j]} - N_{,b}^a t_a^{[i} T^{j]b} + N^c t_c^k T^{a[i} T^{j]b} t_{a,b}^k \\ &\quad + N^c t_{c,b}^{[i} T^{j]b} . \end{aligned} \quad (2.4)$$

The first term on the right hand side of (2.4) can be varied arbitrarily by adding SO(3) rotations to the time derivative of the triad, so  $\Omega_0^{ij}$  in fact can be taken to be an independent gauge variable. We shall see below, in (2.9), that  $\Omega_0^{ij}$  is precisely this arbitrary rotation. The 3-dimensional curvature  ${}^3R_{ab}^{ij}$  is constructed from the three-dimensional rotation coefficients  $\omega_a^{ij}$ .

Variation of the action yields the primary constraints in phase space

$$0 = P_a^i + K_a^i = \tilde{\pi}_i^a = \tilde{P} = \tilde{P}_a = \tilde{P}_i , \quad (2.5)$$

where  $P_a^i$ ,  $\tilde{\pi}_i^a$ ,  $\tilde{P}$ ,  $\tilde{P}_a$ , and  $\tilde{P}_i$  are the momenta conjugate to  $\tilde{T}_i^a$ ,  $K_a^i$ ,  $\tilde{N}$ ,  $N^a$ , and  $\Omega_0^i$ , respectively.

The first two constraints in (2.5) are second class in the sense of Dirac [22,23]; the variables  $P_a^i$  and  $\tilde{\pi}_i^a$  are eliminated in phase space, so that the Dirac bracket becomes

$$\{\tilde{T}_i^a, K_b^j\} = -\delta^3(x-x') \delta_b^a \delta_i^j ,$$

leading to the canonical Hamiltonian density

$$\begin{aligned} \tilde{\mathcal{H}}_c &= -\epsilon^{ijk} \tilde{T}_i^a K_a^j \Omega_0^k + 2N^a \tilde{T}_i^b D_{[a} K_{b]}^i \\ &\quad - \frac{1}{2} \tilde{N} \tilde{T}_i^a \tilde{T}_j^b ({}^3R_{ab}^{ij} + K_a^i K_b^j - K_b^i K_a^j) . \end{aligned}$$

Preservation in time of the remaining primary constraints leads to the secondary constraints

$$\tilde{\mathcal{H}}_i = -\epsilon^{ijk} K_a^j \tilde{T}_k^a = 0 , \quad (2.6)$$

$$\tilde{\mathcal{H}}_a = -2\tilde{T}_i^b D_{[b} K_{a]}^i = 0 , \quad (2.7)$$

$$\tilde{\mathcal{H}}_0 = -\frac{1}{2} \tilde{T}_i^a \tilde{T}_j^b ({}^3R_{ab}^{ij} + K_a^i K_b^j - K_b^i K_a^j) = 0 . \quad (2.8)$$

There are no more constraints. Note that (2.6) is just the condition that  $K_{ab}$  is symmetric. Furthermore, (2.7) can be rewritten in the familiar form

$$D_a K_b^b - D_b K_a^b = 0 .$$

We now turn to the canonical equations of motion. The variation with respect to  $K_a^i$  of  $\int d^3x \tilde{\mathcal{H}}_c$  is straightforward, leading to:

$$\begin{aligned} \dot{\tilde{T}}_i^a &= -\frac{\delta}{\delta K_a^i} \int d^3x' \tilde{\mathcal{H}}'_c \\ &= -\epsilon^{ijk} \tilde{T}_j^a \Omega_0^k + 2D_b N^{[b} \tilde{T}_i^{a]} \\ &\quad + \tilde{N} \tilde{T}_i^a \tilde{T}_j^b K_b^j - \tilde{N} \tilde{T}_i^b \tilde{T}_j^a K_b^j . \end{aligned} \quad (2.9)$$

The variation with respect to  $\tilde{T}_i^a$  requires more work.

We note that the variation of  $\omega_a^{ij}$  must be a tensor, so we can replace ordinary derivatives by covariant derivatives. Also, using the fact that the covariant derivative of the triad is zero, we find that

$$\begin{aligned} \delta\omega_a^{ij} &= g_{ac} \tilde{T}^{b[i} D_b \delta\tilde{T}^{j]c} + \tilde{T}^{b[i} \tilde{t}_c^{j]} \tilde{t}_a^c D_b \delta\tilde{T}_k^c \\ &\quad + \tilde{t}_b^{[i} D_a \delta\tilde{T}^{j]b} + \tilde{t}_c^k \tilde{t}_a^{[i} \tilde{T}^{j]b} D_b \delta\tilde{T}_k^c . \end{aligned}$$

Then an integration by parts in the varied Hamiltonian yields

$$\begin{aligned} \dot{K}_a^i &= -\epsilon^{ijk} K_a^j \Omega_0^k + N^b D_b K_a^i + D_a N^b K_b^i \\ &\quad - \tilde{N} \tilde{T}_j^b ({}^3R_{ab}^{ij} + K_a^i K_b^j - K_b^i K_a^j) \\ &\quad + \tilde{T}_i^b D_a D_b \tilde{N} . \end{aligned}$$

### III. PROJECTABILITY OF GAUGE TRANSFORMATIONS

We first consider triad gauge rotations. Under an infinitesimal rotation with descriptor  $\lambda^i$  the resulting variation of the triad field is

$$\delta_R[\lambda]\tilde{T}_i^a = -\epsilon^{ijk}\lambda^j\tilde{T}_k^a, \quad (3.1)$$

while the variation of  $K_a^i$  is

$$\delta_R[\lambda]K_a^i = -\epsilon^{ijk}\lambda^jK_a^k. \quad (3.2)$$

The Ricci rotation coefficients transform as a connection in the usual manner under this rotation:

$$\delta_R[\lambda]\Omega_0^i = -\partial_0\lambda^i - \epsilon^{ijk}\lambda^j\Omega_0^k. \quad (3.3)$$

The Lagrangian density does not depend on time derivatives of  $N$ ,  $N^a$ , or  $\Omega_0^i$ . Thus, null vectors of the Legendre matrix (the second partial derivative of the Lagrangian density with respect to the velocities) are  $\partial/\partial\dot{N}$ ,  $\partial/\partial\dot{N}^a$ , and  $\partial/\partial\dot{\Omega}_0^i$ ; in other words, projectable symmetry variations under the Legendre transformation may not depend on  $\dot{N}$ ,  $\dot{N}^a$ , or  $\dot{\Omega}_0^i$ , as discussed in [1]. As in the conventional formulation of general relativity, the variations of lapse and shift are not projectable unless the descriptors  $\epsilon^\mu$  of an infinitesimal coordinate transformation  $x^\mu \rightarrow x^\mu - \epsilon^\mu$  depend on the lapse and shift in the following manner:

$$\epsilon^\mu = \delta_a^\mu \xi^a + n^\mu \xi^0, \quad (3.4)$$

where  $n^\mu = (N^{-1}, -N^{-1}N^a)$  is the normal to the constant coordinate time hypersurface, the  $\xi^\mu$  being arbitrary functions.

We must now check whether variations of  $\Omega_0^{ij}$  are projectable under these diffeomorphisms. We represent the variations resulting from infinitesimal perpendicular diffeomorphisms, with descriptors  $\epsilon^\mu = n^\mu \xi^0$ , by  $\delta_{PD}$ , where  $PD$  denotes perpendicular diffeomorphism. (Since variations under diffeomorphisms for which  $\delta x^0 = 0$  are projectable—they are the usual Lie derivatives of spatial vectors under spatial diffeomorphisms—we will not consider them here.) The objective of this calculation is the evaluation of the variation of  $\Omega_0^{ij}$  on-shell, that is, using the equations of motion. In this sense,  $\Omega_0^{ij}$  should not here be considered as an independent variable.

Let us first calculate the variations of the tetrad vectors. We find

$$\begin{aligned} \delta_{PD}[\xi^0]E_0^0 &= -N^{-2}\xi^0 + N^{-2}N^a\xi_{,a}^0, \\ \delta_{PD}[\xi^0]E_i^0 &= 0, \\ \delta_{PD}[\xi^0]E_0^a &= N^{-2}N^a\xi^0 - N^{-2}N^aN^b\xi_{,b}^0 \\ &\quad - N^{-1}\epsilon^{ab}N_{,b}\xi^0 + \epsilon^{ab}\xi_{,b}^0, \\ \delta_{PD}[\xi^0]E_i^a &= \delta_{PD}[\xi^0]T_i^a \\ &= \dot{T}_i^aN^{-1}\xi^0 - T_{i,b}^aN^bN^{-1}\xi^0 \\ &\quad + N_{,b}^aN^{-1}T_i^b\xi^0. \end{aligned}$$

The equation of motion (2.9) is equivalent to

$$\dot{T}_i^a = -\Omega_0^{ij}T_j^a - D_bN^aT_i^b - NT_i^bT_j^aK_b^j.$$

The final expression for the last variation may consequently be rewritten as

$$\delta_{PD}[\xi^0]T_i^a = -\xi^0\Omega_\mu^{ij}n^\mu T_j^a - \xi^0T_i^bT_j^aK_b^j.$$

Since we shall require the result below, we also include here the corresponding variation of the densitized triad:

$$\delta_{PD}[\xi^0]\tilde{T}_i^a = -\xi^0\Omega_\mu^{ij}n^\mu\tilde{T}_j^a - \xi^0T_i^b\tilde{T}_j^aK_b^j + \xi^0T_i^a\tilde{T}_j^bK_b^j. \quad (3.5)$$

Similarly, we find that

$$\delta_{PD}[\xi^0]e_0^0 = \dot{\xi}^0 - N^a\xi_{,a}^0, \quad (3.6)$$

$$\delta_{PD}[\xi^0]e_a^0 = 0, \quad (3.7)$$

$$\begin{aligned} \delta_{PD}[\xi^0]e_0^i &= -N^a\Omega_\mu^{ij}n^\mu\xi^0t_a^j + N^a\xi^0K_a^i \\ &\quad - N_{,a}T^{ai}\xi^0 + T^{ai}\xi_{,a}^0, \end{aligned} \quad (3.8)$$

$$\delta_{PD}[\xi^0]e^i_a = \delta_{PD}[\xi^0]t^i_a = -\xi^0\Omega_\mu^{ij}n^\mu t_a^j + \xi^0K_a^i. \quad (3.9)$$

We shall also require the variation of the densitized lapse  $\tilde{N}$ . Referring to (3.9) to compute  $\delta_{PD}[\xi^0]t$ , we find

$$\delta_{PD}[\xi^0]\tilde{N} = -t^{-1}N\xi^0K_a^iT_i^a + t^{-1}\dot{\xi}^0 - t^{-1}N^a\xi_{,a}^0.$$

Using

$$\dot{t} = tD_aN^a + tNT_i^aK_a^i, \quad {}^3\Gamma_{ba}^b = t^{-1}t_{,a},$$

we find

$$\delta_{PD}[\xi^0]\tilde{N} = \dot{\xi}^0 + \xi^0N_{,a}^a - N^a\xi_{,a}^0. \quad (3.10)$$

Substituting the variations of the tetrad vectors into (2.1), we find

$$\begin{aligned} \delta_{PD}[\xi^0]\Omega_0^{ij} &= 2NN^aK_a^{[i}T^{j]b}(N^{-1}\xi^0)_{,b} \\ &\quad + 2NN_{,a}T^{a[i}T^{j]b}(N^{-1}\xi^0)_{,b} \\ &\quad + (\Omega_\mu^{ij}n^\mu\xi^0)_{,0} + \dot{\Omega}_0^{ij}N^{-1}N^a\xi^0 \\ &\quad - \Omega_{0,a}^{ij}N^{-1}N^a\xi^0. \end{aligned} \quad (3.11)$$

It is noteworthy that the first two terms in this expression appear due to the fact that our tetrad vectors are not true four-vectors (since they are tied to the time foliation). Otherwise this variation agrees with the standard result in Einstein-Yang-Mills theory with an  $SO(3)$  connection [2]. To proceed further we must substitute the time derivative

$$\begin{aligned}\dot{\Omega}_a^{ij} &= D_a \Omega_0^{ij} + 2K_a^{[i} T^{j]b} N_{,b} \\ &\quad + 4NT^{b[i} D_{[a} K_{b]}^{j]} + {}^3R_{ba}^{ij} N^b .\end{aligned}\quad (3.12)$$

We find ultimately that

$$\begin{aligned}\delta_{PD}[\xi^0]\Omega_0^{ij} &= -4\xi^0 N^a D_{[a} K_{b]}^{[i} T^{j]b} \\ &\quad + 2N^b \xi_{,a}^0 K_b^{[i} T^{j]a} + 2N_{,b} \xi_a^0 T^{b[i} T^{j]a} \\ &\quad + (\Omega_\mu^{ij} n^\mu \xi^0)_{,0} + 2\xi^0 n^\mu \Omega_\mu^{[i} \Omega_0^{j]} .\end{aligned}\quad (3.13)$$

We have displayed this result in a form, (3.13), in which it is manifest that the variation is not projectable: The next to the last term contains time derivatives of all three gauge variables; but the resolution is manifest as well. Referring to (3.3) we see that the last two terms are precisely a gauge rotation of the  $\text{SO}(3)$  connection components  $\Omega_\mu^{ij}$  with a descriptor  $\lambda^i = -\Omega_\mu^i n^\mu \xi^0$ . To obtain a projectable variation we must accompany the perpendicular diffeomorphism with an  $\text{SO}(3)$  gauge rotation with minus this descriptor.

This is exactly the form of the gauge transformation which must be added to diffeomorphisms in the Einstein-Yang-Mills case to produce a projectable variation under the Legendre transformation, as shown in [2]. The demonstration that this is the required addition here was complicated somewhat by the fact that the connection is not a one-form under diffeomorphisms which alter the time foliation. As mentioned above this is due to the fact that the tetrad vectors in our  $3+1$  decomposition are not four-vectors.

We close this section with the variation of  $\Omega_0^i$  under spatial diffeomorphisms, with descriptor  $\epsilon^\mu = \delta_a^\mu \xi^a$ . We will represent the variation by  $\delta_{SD}[\vec{\xi}]$  where SD denotes spatial diffeomorphism. This variation is indeed the Lie derivative since the tetrads do transform as manifest four-vectors under infinitesimal transformations which do not alter the spacetime foliation:

$$\delta_{SD}[\vec{\xi}]\Omega_0^i = \Omega_{0,a}^i \xi^a + \Omega_a^i \xi_{,0}^a .\quad (3.14)$$

We wish to obtain the on-shell variation which we will later on compare with variations generated by symmetry generators which we construct below. For this purpose it is convenient to rewrite (3.14) as

$$\begin{aligned}\delta_{SD}[\vec{\xi}]\Omega_0^i &= (D_a \Omega_0^i - \dot{\Omega}_a^i) \xi^a + (\Omega_a^i \xi^a)_{,0} - \Omega_a^{ij} \xi^a \Omega_0^j \\ &= -\epsilon^{ijk} \xi^a (K_a^j T^{bk} N_{,b} + 2NT^{bj} D_{[a} K_{b]}^k) \\ &\quad - {}^3R_{ba}^i N^b \xi^a + (\xi^a \omega_a^i)_{,0} + \epsilon^{ijk} \xi^a \omega_a^j \Omega_0^k .\end{aligned}\quad (3.15)$$

where in the second equality we used the time derivative (3.12).

#### IV. GENERATORS, VARIATIONS, AND LIE ALGEBRA

Our next task is to construct the generators of gauge transformations in order to verify that the phase space

calculations reproduce the above configuration-velocity space results. For this purpose we first introduce generators associated with our secondary constraints:

$$\begin{aligned}R[\xi] &:= \int d^3x \xi^i \tilde{\mathcal{H}}_i , \\ V[\vec{\xi}] &:= \int d^3x \xi^a \tilde{\mathcal{H}}_a , \\ S[\xi^0] &:= \int d^3x \xi^0 \tilde{\mathcal{H}}_0 .\end{aligned}$$

We find that  $R[\xi]$  generates an  $\text{SO}(3)$  gauge rotation, so we have, for example,

$$\begin{aligned}\delta_R[\xi] \tilde{T}_i^a &= \{ \tilde{T}_i^a, -\epsilon_{jkl} \int d^3x' \xi'^j K_b'^k \tilde{T}_l'^b \} \\ &= -\epsilon_{ijl} \xi^j \tilde{T}_l^a .\end{aligned}$$

$V[\vec{\xi}]$  generates a spatial diffeomorphism plus  $\text{SO}(3)$  gauge rotation:

$$\begin{aligned}\delta_V[\vec{\xi}] \tilde{T}_i^a &= \{ \tilde{T}_i^a, -2 \int d^3x' \xi'^b \tilde{T}_i'^c D_{[c} K_{b]}'^a \} \\ &= 2D_b(\xi^b \tilde{T}_i^a) \\ &= \mathcal{L}_{\vec{\xi}} \tilde{T}_i^a + \delta_R[\xi^b \omega_b] \tilde{T}_i^a .\end{aligned}$$

It is convenient to define a related generator  $D[\vec{\xi}]$  which generates a pure spatial diffeomorphism:

$$D[\vec{\xi}] := \int d^3x \xi^a \tilde{\mathcal{G}}_a ,$$

where

$$\tilde{\mathcal{G}}_a := \tilde{\mathcal{H}}_a - \omega_a^i \tilde{\mathcal{H}}_i .$$

$S[\xi^0]$  generates a space-time diffeomorphism, with descriptor  $\xi^0 = t\xi^0$ , plus a gauge rotation (neither of which by itself is projectable). So, for example,

$$\begin{aligned}\delta_S[\xi^0] \tilde{T}_i^a &= \delta_{PD}[t\xi^0] \tilde{T}_i^a + \delta_R[t\xi^0 \omega_\mu n^\mu] \tilde{T}_i^a \\ &= -\xi^0 \tilde{T}_i^b \tilde{T}_j^a K_b^j + \xi^0 \tilde{T}_i^a \tilde{T}_j^b K_b^j ,\end{aligned}$$

where in the last line we used (3.5).

We now turn to the calculation of the complete Lie algebra in configuration-velocity space. It would be straightforward to calculate the algebra from the calculable action of the infinitesimal group elements on the generators. The only Poisson bracket we will not calculate in this manner is the bracket of  $S[\xi^0]$  with  $S[\eta^0]$ ,

simply because it would be tedious, invoking time derivatives of the triad vectors, the curvature, and the extrinsic curvature.

First, a gauge rotation of  $\tilde{\mathcal{H}}_i$  yields

$$\{R[\xi], R[\eta]\} = -R[[\xi, \eta]] ,$$

where we define the commutator bracket as

$$[\xi, \eta]^i := \epsilon^{ijk} \xi^j \eta^k .$$

In the following expressions, we will also use the Lie bracket

$$[\vec{\xi}, \vec{\eta}]^a := \xi^b \eta_{,b}^a - \eta^b \xi_{,b}^a .$$

The remaining brackets are

$$\begin{aligned} \{R[\xi], D[\vec{\eta}]\} &= \int d^3x \xi^i \mathcal{L}_{\vec{\eta}} \tilde{\mathcal{H}}_i \\ &= - \int d^3x \mathcal{L}_{\vec{\eta}} \xi^i \tilde{\mathcal{H}}_i \\ &= -R[\mathcal{L}_{\vec{\eta}} \xi] , \end{aligned}$$

$$\begin{aligned} \{D[\vec{\xi}], D[\vec{\eta}]\} &= \int d^3x \xi^a \mathcal{L}_{\vec{\eta}} \tilde{\mathcal{G}}_a \\ &= - \int d^3x \mathcal{L}_{\vec{\eta}} \xi^a \tilde{\mathcal{G}}_a \\ &= -D[\mathcal{L}_{\vec{\eta}} \vec{\xi}] = D[[\vec{\xi}, \vec{\eta}]] , \end{aligned}$$

$$\begin{aligned} \{S[\xi^0], D[\vec{\eta}]\} &= \int d^3x \xi^0 \mathcal{L}_{\vec{\eta}} \tilde{\mathcal{H}}_0 \\ &= - \int d^3x \mathcal{L}_{\vec{\eta}} \xi^0 \tilde{\mathcal{H}}_0 \\ &= -S[\mathcal{L}_{\vec{\eta}} \xi] , \end{aligned}$$

$$\{S[\xi^0], R[\eta]\} = 0 ,$$

$$\{V[\vec{\xi}], R[\eta]\} = 0 .$$

The last two brackets result from the fact that  $\tilde{\mathcal{H}}_0$  and  $\tilde{\mathcal{H}}_a$  are gauge rotation scalars. Finally, direct calculation yields

$$\{S[\xi^0], S[\eta^0]\} = V[\vec{\zeta}] - R[[\xi, {}_a T^a, \eta, {}_b T^b]] ,$$

where

$$\zeta^a := (\xi \partial_b \eta - \eta \partial_b \xi) \tilde{\epsilon}^{ab} .$$

From these brackets we next determine the brackets among the  $R$ ,  $V$ , and  $S$  generators alone. We find

$$\begin{aligned} \{V[\vec{\xi}], V[\vec{\eta}]\} &= \{D[\vec{\xi}] + R[\xi^a \omega_a], D[\vec{\eta}] + R[\eta^b \omega_b]\} \\ &= V[[\vec{\xi}, \vec{\eta}]] - R[{}^3R_{ab} \xi^a \eta^b] . \end{aligned}$$

The remaining bracket is

$$\begin{aligned} \{S[\xi^0], V[\vec{\eta}]\} &= \{S[\xi^0], D[\vec{\eta}] + R[\xi^a \omega_a]\} \\ &= -S[\mathcal{L}_{\vec{\eta}} \xi^0] - R[\eta^a (\delta_{PD}[t \xi^0] + \delta_R[\Omega_\mu n^\mu t \xi^0]) \omega_a] , \end{aligned}$$

where

$$\begin{aligned} &(\delta_{PD}[t \xi^0] + \delta_R[\Omega_\mu n^\mu t \xi^0]) \omega_a^i \\ &= (K_a^j T^{bk} D_b(\xi^0) + 2 \tilde{T}_j^b \xi^0 D_{[a} K_{b]}^k) \epsilon^{ijk} . \end{aligned}$$

## V. COMPLETE SYMMETRY GENERATORS

The canonical Hamiltonian in terms of the generators takes the simple form

$$H = \int d^3x N^A \mathcal{H}_A =: N^A \mathcal{H}_A ,$$

where we define

$$N^A := \{N, N^a, -\Omega_0^i\} , \quad \mathcal{H}_A = \{\tilde{\mathcal{H}}_0, \tilde{\mathcal{H}}_a, \tilde{\mathcal{H}}_i\} ,$$

and where spatial integrations over corresponding repeated primed indices are assumed. It was shown in [1] that the complete symmetry generators then take the form

$$G(t) = \xi^A G_A^{(0)} + \dot{\xi}^A G_A^{(1)} . \quad (5.1)$$

The descriptors  $\xi^A$  are arbitrary functions. The simplest choice for the  $G_A^{(1)}$  are the primary constraints  $P_A$ ,

$$P_A := \{\tilde{P}, \tilde{P}_a, -\tilde{P}_i\} ,$$

with the result that

$$G[\xi] = P_A \dot{\xi}^A + (\mathcal{H}_A + P_{C''} N^{B'} \mathcal{C}_{AB'}^{C''}) \xi^A ,$$

where the structure functions are:

$$\{\mathcal{H}_A, \mathcal{H}_{B'}\} =: \mathcal{C}_{AB'}^{C''} \mathcal{H}_{C''} . \quad (5.2)$$

Using the brackets calculated in the previous section we read off the following non-vanishing structure functions: From these brackets we next determine the brackets among the  $R$ ,  $V$ , and  $S$  generators alone. We find

$$\begin{aligned}
C_{0'0''}^a &= \tilde{\epsilon}^{ab} \left( -\delta^3(x-x') \partial_b'' \delta^3(x-x'') \right. \\
&\quad \left. + \delta^3(x-x'') \partial_b' \delta^3(x-x') \right) , \\
C_{b'c''}^a &= -\delta^3(x-x') \partial_b'' \delta^3(x-x'') \delta_c^a \\
&\quad + \delta^3(x-x'') \partial_c' \delta^3(x-x') \delta_b^a , \\
C_{0'a''}^0 &= \delta^3(x-x'') \partial_a' \delta^3(x-x') \\
&\quad - \delta^3(x-x') \partial_a'' \delta^3(x-x'') , \\
C_{j'k''}^i &= -\epsilon^{ijk} \delta^3(x-x') \delta^3(x-x'') , \\
C_{0'a''}^i &= \epsilon^{ijk} \left( K_a^j T^{bk} t' \partial_b' \delta^3(x-x') \delta^3(x-x'') \right. \\
&\quad \left. - 2D_{[a} K_{b]}^i \tilde{T}_j^b \delta^3(x-x') \delta^3(x-x'') \right) , \\
C_{a'b''}^i &= -{}^3R_{ab}^i \delta^3(x-x') \delta^3(x-x'') , \\
C_{0'0''}^i &= -\epsilon^{ijk} t' t'' \left( \partial_a' \partial_b'' (T'^{aj} T''^{bk} \delta^3(x-x') \delta^3(x-x'')) \right) .
\end{aligned}$$

With the use of the structure functions derived above, we obtain the following generators, denoted by  $G[\xi]$ ,  $G[\eta]$ , and  $G[\zeta^0]$ . These are, respectively, the gauge, spatial diffeomorphisms, and perpendicular diffeomorphisms (plus associated gauge rotations in the last two cases):

$$\begin{aligned}
G[\xi] &= \int d^3x \left( -\tilde{P}_i \xi^i + \tilde{\mathcal{H}}_i \xi^i - \epsilon_{ijk} \xi^j \Omega_0^k \tilde{P}_i \right) , \\
G[\eta] &= \int d^3x \left( \tilde{P}_a \eta^a + {}^3R_{ab}^i \tilde{P}_i \eta^a N^b \right. \\
&\quad \left. - \epsilon^{ijk} ((t\tilde{N})_{,b} \eta^a K_a^j \tilde{T}^{bk} \tilde{P}_i \right. \\
&\quad \left. + 2D_{[a} K_{b]}^k \tilde{T}_j^b N \eta^a \tilde{P}_i) - \tilde{N} \tilde{P}_{,a}^a + \tilde{N}_{,a} \tilde{P}^a \right. \\
&\quad \left. + N_{,a} \tilde{P}^a \eta^b - N^b \eta_{,b} \tilde{P}_a + \eta^a \tilde{\mathcal{H}}_a \right) , \\
G[\zeta^0] &= \int d^3x \left( \tilde{P} \zeta^0 + \tilde{N}_{,b} \tilde{P}_a \zeta^0 \tilde{\epsilon}^{ab} + \zeta^0 \tilde{\mathcal{H}}_0 \right. \\
&\quad \left. - \tilde{N} \tilde{P}_{,a} \zeta^0 \tilde{\epsilon}^{ab} - N^a \tilde{P}_{,a} \zeta^0 + N_{,a}^a \tilde{P} \zeta^0 \right. \\
&\quad \left. + \left( (t\zeta^0)_{,a} (t\tilde{N})_{,b} \tilde{T}^{aj} \tilde{T}^{bk} + (t\zeta^0)_{,b} K_a^j T^{bk} N^a \right. \right. \\
&\quad \left. \left. + 2\zeta^0 D_{[a} K_{b]}^k \tilde{T}_j^b N^a \right) \epsilon^{ijk} \tilde{P}_i \right) .
\end{aligned}$$

These generators do indeed generate all of the correct symmetry variations, including the variations of the gauge variables  $\tilde{N}$  and  $\Omega_0^i$  displayed in (3.10), (3.13), and (3.15).

## VI. CONCLUSIONS

We have constructed explicitly the full set of generators of projectable triad rotations, spatial diffeomorphisms, and perpendicular diffeomorphisms in vacuum General Relativity in a real triad formalism. The perpendicular diffeomorphisms are not by themselves projectable

(generated by functions which can be pulled back to the original configuration-velocity space). They must be accompanied by triad rotations which themselves depend on metric, triad, and extrinsic curvature. The generators act in a phase space which includes as variables the gauge functions  $N$ ,  $N^a$ , and  $\Omega_0^i$ . The group algebra can be deduced from the action of the transformation group in configuration-velocity space. We did so in this paper, except for  $\{S[\xi^0], S[\eta^0]\}$ , simply because direct calculation of the Poisson bracket was more efficient in this case. The manifold on which this enlarged symmetry group acts must be interpreted to be the set of all solution trajectories, and the group elements contain a compulsory dependence on these solutions. (The pullback from phase space to configuration-velocity space of variations of arbitrary trajectories does not generally yield Noether symmetries. However, the pullbacks of configuration variables alone do produce Noether symmetries. See [2] and [21] for further details.)

Our results for the symmetry variations and corresponding generators differ technically somewhat from the results in Einstein-Yang-Mills theories, found in [2]. In this paper a  $3+1$  decomposition of the Ricci rotation coefficients destroys the manifest covariance of these coefficients. This lack of manifest covariance is to be contrasted with the Yang-Mills case in which the connection is a four-dimensional one-form. The result is a commutator algebra which differs from the Einstein-Yang-Mills algebra.

Our analysis was undertaken in part to compare and contrast these results with the symmetry properties of the real sector of Ashtekar's formulation of general relativity [15–17]. In fact, many of the computations performed in this paper are considerably simplified by working with the canonically transformed complex variables in Ashtekar's approach. It does turn out, of course, that the underlying dynamics in the Ashtekar theory, when restricted to real triads, does coincide with the real triad dynamics presented in this paper. Indeed, the symmetry variations in the complex Ashtekar formalism do preserve the reality of real triads, and they coincide with the symmetries presented here. It does turn out, however, that the functional form of the gauge rotation required to obtain projectability of perpendicular diffeomorphisms plus gauge is altered. An additional complex triad rotation is required. These results will be presented in a forthcoming paper [20]. The issue of projectability is intimately related to the problem of preservation of reality conditions under time evolution.

The results we have obtained in this and preceding papers represent a significant conceptual and technical advancement. In the context of the Dirac-Bergmann constraint formalism, the conventional wisdom is that first class phase space generators are to be interpreted as generators of gauge transformations [5, 22, 23]. Yet this notion has been rejected historically due to the presence

of dynamical fields in the Poisson bracket algebra, leading to what some call an open algebra; see, for example, [17]. In these papers we require that Lagrangian Noether symmetry variations be projectable to variations in phase space. We find—amazingly—that the previously known Hamiltonian constraints generate these projected phase space gauge variations. The full four-dimensional diffeomorphism group is only implementable on shell, that is, on solutions of the equations of motion. We conclude that the gauge group present in configuration-velocity space is projectable to phase space. The original full Lagrangian symmetry is therefore retained in the Hamiltonian formalism.

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